Stochastic heat equation with rough multiplicative noise

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Consider the one-dimensional stochastic heat equation on $\mathbb{R}$:

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\frac{\partial^2 W}{\partial x \partial t}, \qquad (1)$$

with initial condition u_0, where $\kappa > 0$ is a fixed parameter.

- The noise $W = \{W(t,x), t \geq 0, x \in \mathbb{R}\}$ is a centered Gaussian process with covariance given by

$$E(W(s,x)W(t,y)) = (s \wedge t)\frac{1}{2}(|x|^{2H} + |y|^{2H} - |x-y|^{2H})$$

  with $\frac{1}{4} < H < \frac{1}{2}$. That is, W is a Brownian motion in time and a *fractional Brownian motion* with Hurst parameter H in space.

- The covariance of $\frac{\partial^2 W}{\partial x \partial t}$ equals to $H(2H-1)\delta_0(t-s)|x-y|^{2H-2}$, is NOT locally integrable in space when $H < \frac{1}{2}$.

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Stochastic integration with respect to W

Integration of deterministic functions:

- Let $\mathcal{H}$ be the closure of $\mathcal{D}((0,\infty)\times\mathbb{R})$ under the semi-norm

$$\|f\|_{\mathcal{H}}^2 = c_{1,H}\int_0^\infty \int_{\mathbb{R}^2} |f(s,x+y)-f(s,x)|^2|y|^{2H-2}dxdyds,$$

where $c_{1,H} = H(1-2H)/2$.

- The space $\mathcal{H}$ is isometric to the Gaussian space spanned by W: the mapping $\mathbf{1}_{[0,t]\times[0,x]} \mapsto W(\mathbf{1}_{[0,t]\times[0,x]}) = W(t,x)$ can be extended to $\mathcal{H}$, and $E(W(f)^2) = \|f\|_{\mathcal{H}}^2$.
- Using the Fourier transform in the space variable yields

$$E(W(f)^2) = c_{2,H}\int_0^\infty \int_{\mathbb{R}} |\mathcal{F}f(s,\xi)|^2|\xi|^{1-2H}d\xi ds,$$

where $c_{2,H} = \frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H)$.

- $\mu(d\xi) = |\xi|^{1-2H}d\xi$ is the *spatial spectral measure*. Its Fourier transform is not a function when $H < \frac{1}{2}$.

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Integration of predictable processes:

- Let $\mathcal{F}_t$ be the filtration generated by W up to time t. An elementary predictable process u is given by:

$$u(s,x) = \sum_{i=1}^{n} X_i \mathbf{1}_{(a_i,b_i]}(s)\varphi_i(x),$$

where $0 \leq a_1 < b_1 < \cdots < a_n < b_n < \infty$, $\varphi_i \in \mathcal{D}(\mathbb{R})$ and X_i is $\mathcal{F}_{a_i}$-measurable and bounded, for $i = 1, \ldots, n$.

- For such process we define

$$\int_0^\infty \int_{\mathbb{R}} u(s,x) W(ds,dx) = \sum_{i=1}^{n} X_i W(\mathbf{1}_{(a_i,b_i]} \otimes \varphi_i).$$

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Proposition

Let Λ_H be the space of predictable processes g such that $E[\|g\|_{\mathcal{H}}^2] < \infty$. Then,

(i) *The space of elementary predictable processes is dense in Λ_H.*

(ii) *The stochastic integral can be extended to Λ_H, and we have:*

$$E\left(\left|\int_0^\infty \int_{\mathbb{R}} g(s,x)\, W(ds,dx)\right|^2\right) = E\left[\|g\|_{\mathcal{H}}^2\right].$$

Mild solution

- We denote by $p_t(x) = \frac{1}{\sqrt{2\pi\kappa t}} e^{-x^2/2\kappa t}$ the heat kernel.

Definition

Let $u = \{u(t,x), t \geq 0, x \in \mathbb{R}\}$ be a real-valued predictable stochastic process such that for all $t \geq 0$ and $x \in \mathbb{R}$ the process

$$\{p_{t-s}(x-y)u(s,y)\mathbf{1}_{[0,t]}(s), 0 \leq s \leq t, y \in \mathbb{R}\}$$

is an element of Λ_H. We say that u is a mild solution of (1) if for all $t \geq 0$ and $x \in \mathbb{R}$ we have:

$$u(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy).$$

A stochastic Young inequality I

- For $p \geq 1$ define

$$\|u\|^2_{\mathcal{X}^p_T} = \sup_{\substack{t\in[0,T]\\ x\in\mathbb{R}}} \left(\|u(t,x)\|^2_{L^p(\Omega)} + \int_{\mathbb{R}} \frac{\|u(t,x)-u(t,x+y)\|^2_{L^p(\Omega)}}{|y|^{2-2H}} dy \right).$$

- Define the stochastic convolution of a predictable process Z as

$$(p * ZW)(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)Z(s,y)W(ds,dy).$$

Proposition

For any $p \geq 2$,

$$\|p * ZW)\|_{\mathcal{X}^p_T} \leq C_{T,H}\sqrt{p}\|Z\|_{\mathcal{X}^p_T}. \tag{2}$$

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Sketch of the proof:
(i) Using Burkholder's inequality,

$$\begin{aligned} & \|(p * ZW)(t,x)\|_{L^p(\Omega)} \\ \leq \quad & C\sqrt{p}\Big\| \int_0^t \int_{\mathbb{R}^2} (p_{t-s}(x-y)Z(s,y) - p_{t-s}(x-y-z)Z(s,y+z))^2 \\ & \quad \times |z|^{2H-2} dydzds \Big\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)}. \end{aligned}$$

(ii) The estimate follows using Minkowski's inequality and Fourier transform arguments.
(iii) In the same way we handle the term

$$\int_{\mathbb{R}} \frac{\|(p * ZW)(t,x) - (p * ZW)(t,x+y)\|_{L^p(\Omega)}}{|y|^{2-2H}} dy.$$

Existence and uniqueness in the affine case

Theorem (Balan, Jolis, Quer-Sardanyons '15)

Suppose that $\frac{1}{4} < H < \frac{1}{2}$ and

(i) *u_0 is bounded and H-Hölder continuous.*

(ii) *$\sigma(u) = au + b$.*

Then, there is a unique mild solution to equation (1) in $\mathcal{X}_T^2$. Moreover, the solution is $L^2(\Omega)$-continuous and for each $p \geq 2$ it belongs to $\mathcal{X}_T^p$.

- If σ is an affine function, then, if u and v are two solutions,

$$\sigma(u(s,y)) - \sigma(v(s,y) - \sigma(u(s,y+z)) + \sigma(v(s,y+z))$$
$$= a[u(s,y) - v(s,y) - u(s,y+z) + v(s,y+z)].$$

A stochastic Young inequality II

- For $p \geq 1$ define

$$\|u\|^2_{\mathcal{Z}^p_T} = \sup_{t\in[0,T]} \left(\|u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} + \int_{\mathbb{R}} \frac{\|u(t,\cdot) - u(t,\cdot+y)\|^2_{L^p(\Omega\times\mathbb{R})}}{|y|^{2-2H}} dy \right).$$

Proposition

$$E\left(\sup_{\substack{t\in[0,T]\\ x\in\mathbb{R}}} \int_{\mathbb{R}} \frac{|p * ZW(t,x) - p * ZW(t,x+y)|^2}{|y|^{2-2H}} dy \right)^{\frac{p}{2}} \leq C_{T,H,p} \|Z\|_{\mathcal{Z}^p_T} \quad (3)$$

- The proof is based on the convolution argument as in Gyongy-N. '99.

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Uniqueness of solutions

Theorem

Suppose that $\frac{1}{4} < H < \frac{1}{2}$ *and:*

(i) $u_0 \in L^p(\mathbb{R})$ *for* $p > \frac{6}{4H-1}$.

(ii) σ *has a Lipschitz derivative and* $\sigma(0) = 0$.

Then, if u *and* v *are two solutions in the space* $\mathcal{Z}_T^p$, *for each* t, x, $u(t,x) = v(t,x)$ *almost surely.*

Main ingredients of the proof:

- Given two solutions u, v, we need to estimate a double increment of the form

$$\begin{aligned}&|\sigma(u(s,y)) - \sigma(v(s,y) - \sigma(u(s,y+z)) + \sigma(v(s,y+z))| \\ &\leq C|u(s,y) - v(s,y) - u(s,y+z) + v(s,y+z)| \\ &+ C|u(s,y) - v(s,y)|[|u(s,y) - u(s,y+z)| + |v(s,y) - v(s,y+z)|].\end{aligned}$$

- To handle the product term, we need a stopping time argument:

$$T_k = \inf\left\{0 \leq t \leq T : \sup_{0\leq s\leq t, y\in\mathbb{R}} \int_{\mathbb{R}} |u(s,y) - u(s,y+z)|^2 |z|^{2H-2} dz \geq k\right\}$$

- Using inequality (3) we can show that $T_k \uparrow \infty$, a.s. as $k \to \infty$.

Existence of solutions

Theorem

Suppose that σ satisfies (ii) and u_0 satisfies:

(i') *For $p > \frac{6}{4H-1}$, $u_0 \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^H(\mathbb{R})$ and $\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + y)\|_{L^p(\mathbb{R})} |y|^{2H-2} dy < \infty$.*

Then there exists a unique solution in $\mathcal{Z}_T^p \cap \mathcal{X}_T^p$.

- The proof uses a compactness argument on the set of probabilities in space of continuous functions $f \in C([0,T] \times \mathbb{R})$ such that $(t,x) \mapsto \int_{|y|\le 1} |f(t,x+y) - f(t,x)||y|^{2H-2} dy$ is finite and continuous and for all $R > 0$

$$\lim_{z \downarrow 0} \sup_{\substack{t \in [0,T] \\ x \in [-R,R]}} \int_{|y|\le 1} |f(t,x+z) - f(t,x) - f(t,x+y) + f(t,x+z+y)||y|^{2H-2} dy = 0.$$

Moment estimates

The solution satisfies

$$\boxed{\sup_{x\in\mathbb{R}} \|u(t,x)\|_{L^p(\Omega)} \le C_{u_0} \exp\left(Ctp^{\frac{1}{H}} \kappa^{1-\frac{1}{H}} \|\sigma\|_{Lip}^{\frac{2}{H}} \right)}$$

and

$$\boxed{Eu(t,x)^2 \ge C \frac{|p_t u_0(x)|^3}{\|u_0\|_{L^\infty}} \exp\left(Ct\kappa^{1-\frac{1}{H}} \sigma_*^{\frac{2}{H}} \right)},$$

where $\sigma_* = \inf_{u\in\mathbb{R}} \frac{|\sigma(u)|}{|u|}$.

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where $\sigma_* = \inf_{u\in\mathbb{R}} \frac{|\sigma(u)|}{|u|}$.

- The upper bound follows from

$$\boxed{\|p * ZW)\|_{\mathcal{X}^p_{\theta,\epsilon}} \le C_0\sqrt{p}\|Z\|_{\mathcal{X}^p_{\theta,\epsilon}} \left(\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}} + \epsilon^{-1}\kappa^{-\frac{1}{4}}\theta^{-\frac{1}{4}} + \epsilon\kappa^{H-\frac{3}{4}}\theta^{\frac{1}{4}-H}\right),}$$

where for any $p \ge 2$, $\theta, \epsilon > 0$,

$$\|u\|_{\mathcal{X}^p_{\theta,\epsilon}} = \sup_{\substack{t\ge 0\\ x\in\mathbb{R}}} e^{-\theta t}\|u(t,x)\|_{L^p(\Omega)} + \epsilon \sup_{\substack{t\ge 0\\ x\in\mathbb{R}}} e^{-\theta t}\left(\int_{\mathbb{R}} \|u(t,x+y)-u(t,x)\|^2_{L^p(\Omega)}|y|^{2H-2}dy\right)^{\frac{1}{2}}.$$

- The lower bound follows from the Sobolev embedding inequality

$$\int_{\mathbb{R}^2} |g(x+y)-g(x)|^2|y|^{2H-2}dydx \ge c\|g\|^2_{L^{\frac{1}{H}}(\mathbb{R})}.$$

- The upper bound follows from

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where for any $p \geq 2$, $\theta, \epsilon > 0$,

$$\begin{aligned} \|u\|_{\mathcal{X}^p_{\theta,\epsilon}} &= \sup_{\substack{t\geq 0\\ x\in\mathbb{R}}} e^{-\theta t}\|u(t,x)\|_{L^p(\Omega)} \\ &+\epsilon \sup_{\substack{t\geq 0\\ x\in\mathbb{R}}} e^{-\theta t}\left(\int_{\mathbb{R}} \|u(t,x+y)-u(t,x)\|^2_{L^p(\Omega)}|y|^{2H-2}dy\right)^{\frac{1}{2}}. \end{aligned}$$

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Parabolic Anderson model

Suppose $\sigma(u) = u$, that is

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 W}{\partial x \partial t}. \tag{4}$$

- The random field $v = \log u$ satisfies formally the KPZ equation:

$$\frac{\partial v}{\partial t} = \frac{\kappa}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}v\frac{\partial v}{\partial x} + \frac{\partial^2 W}{\partial x \partial t}.$$

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Wiener chaos expansion

- For any fixed (t, x) the random variable $u(t, x)$ admits the following Wiener chaos expansion

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, x)),$$

where for each (t, x), $f_n(\cdot, t, x)$ is a symmetric element in $\mathcal{H}^{\otimes n}$.

- Taking into account that the Itô and Skorohod's integrals coincide for processes in Λ_H and using an iteration procedure, one can find

$$f_n(s_1, x_1, \ldots, s_n, x_n, t, x) = \frac{1}{n!} p_{t-s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}) p_{s_{\sigma(1)}} u_0(x_{\sigma(1)}),$$

where σ denotes the permutation of $\{1, 2, \ldots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$.

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where σ denotes the permutation of $\{1,2,\dots,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$.

Theorem

Assume $\frac{1}{4} < H < \frac{1}{2}$. *Suppose that for any* $a > 0$,

$$\int_{\mathbb{R}} e^{-ax^2} |u_0(x)| dx < \infty.$$

Then, there is a unique mild solution to equation (4).

Sketch of the proof:

Suppose $u_0 = 1$.

- It suffices to show that

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \leq \sum_{n=0}^{\infty} \frac{c^n t^{nH} \kappa^{n(H-1)}}{\Gamma(nH+1)} < \infty.$$

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- Using

$$\mathcal{F} f_n(s_1, \xi_1, \ldots, s_n, \xi_n, t, x) = \frac{c_{2,H}^n}{n!} \prod_{i=1}^n e^{-\frac{\kappa}{2}(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_{\sigma(i)} + \cdots + \xi_{\sigma(1)}|^2} \times e^{-ix(\xi_{\sigma(i)} + \cdots + \xi_{\sigma(i)})},$$

we obtain

$$\|f_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} = \frac{c_{2,H}^{2n}}{(n!)^2} \int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_i + \cdots + \xi_1|^2} |\xi_i|^{1-2H} \, d\xi ds$$

$$= \frac{c_{2,H}^{2n}}{(n!)^2} \int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})\eta_i^2} |\eta_i - \eta_{i-1}|^{1-2H} d\eta ds$$

$$\leq \frac{c_{2,H}^{2n}}{(n!)^2} \int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})\eta_i^2} \left(|\eta_i|^{1-2H} + |\eta_{i-1}|^{1-2H}\right) d\eta ds.$$

Then, we use

$$\int_{\mathbb{R}} e^{-\kappa s \eta^2} |\eta|^{2-4H} d\eta = c s^{-\frac{1}{2}(3-4H)},$$

and $\frac{3-4H}{2} < 1$ if and only if $H > \frac{1}{4}$.

Moment bounds

Theorem

Let $\frac{1}{4} < H < \frac{1}{2}$, and consider the solution u to equation (4) with $u_0 = 1$. Let $p \geq 2$ be an integer. Then

$$\exp(c_1 t p^{1+\frac{1}{H}} \kappa^{1-\frac{1}{H}}) \leq E(u(t,x)^p) \leq \exp\left(c_2 t p^{1+\frac{1}{H}} \kappa^{1-\frac{1}{H}}\right).$$

- These bounds coincide with those obtained in the case $H \geq \frac{1}{2}$ (Khoshnevisan et al.), and also with the upper bound in the case of a general σ.

Proof of the upper bound:

- It follows from the hypercontractivity property

$$\|I_n(f_n(\cdot,t,x))\|_{L^p(\Omega)} \leq (p-1)^{\frac{n}{2}} \|I_n(f_n(\cdot,t,x))\|_{L^2(\Omega)},$$

and the estimate

$$\|I_n(f_n(\cdot,t,x))\|_{L^2(\Omega)} \leq \frac{c^{\frac{n}{2}} t^{\frac{nH}{2}} \kappa^{\frac{n(H-1)}{2}}}{\left[\Gamma(nH+1)\right]^{\frac{1}{2}}},$$

which yields

$$\|u(t,x)\|_{L^p(\Omega)} \leq \sum_{n=0}^{\infty} \frac{c^{\frac{n}{2}} p^{\frac{n}{2}} t^{\frac{nH}{2}} \kappa^{\frac{n(H-1)}{2}}}{\left(\Gamma(nH+1)\right)^{\frac{1}{2}}} \leq \exp\left(c_2 p^{\frac{1}{H}} t \kappa^{1-\frac{1}{H}}\right).$$

Proof of the lower bound:

- We introduce an approximation of the noise given by

$$W^\varepsilon(\varphi) = \int_0^t \int_{\mathbb{R}} [p_\varepsilon * \varphi](s,x) W(ds,dy).$$

- Let u^ε be the solution to the approximate equation

$$u^\varepsilon(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) u^\varepsilon(s,y)\, W^\varepsilon(ds,dy),$$

Proposition (Feynman-Kac formula for the moments)

For each $p \geq 2$, $E[u(t,x)^p] = \lim_{\varepsilon \downarrow 0} E[u^\varepsilon(t,x)^p]$, and

$$E[u^\varepsilon(t,x)^p] = E_B\left[\exp\left(c_{2,H} \sum_{1 \leq i < j \leq p} \int_0^t \int_{\mathbb{R}} e^{-\varepsilon\xi^2 + i\xi(B_r^i - B_r^j)} |\xi|^{1-2H} d\xi dr\right)\right],$$

where B is a p-dimensional Brownian motion independent of W.

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where B *is a p-dimensional Brownian motion independent of* W.

- *Step 1:* Include the diagonal elements:

$$E[u^\varepsilon(t,x)^p] = E_B\left[\exp\left(c\int_0^t\int_{\mathbb{R}} e^{-\varepsilon\xi^2}\left|\sum_{j=1}^p e^{i\xi B_r^j}\right|^2 |\xi|^{1-2H} d\xi dr - cpt\varepsilon^{H-1}\right)\right],$$

- *Step 2:* Reduce the expectation to the event:

$$A_\varepsilon = \left\{\sup_{1\le j\le p}\sup_{0\le r\le t}|B_r^j| \le \frac{\pi}{3}\varepsilon^{\frac{1-H}{2}}\right\},$$

which satisfies $P(A_\varepsilon) \ge ce^{-c\varepsilon^{H-1}pt}$.

On A_ε, assuming $|\xi| \le \varepsilon^{\frac{H-1}{2}}$, we have $|\sum_{j=1}^p e^{iB_r^j\xi}| \ge cp$.

- *Step 3:* As a consequence

$$E[u^\varepsilon(t,x)^p] \geq \exp\left(cp^2 t \varepsilon^{-(1-H)^2} - cpt\varepsilon^{H-1} \right).$$

- *Step 4:* Choosing $\varepsilon = cp^{\frac{1}{H(H-1)}}$ and using

$$E[u^\varepsilon(t,x)^p] \leq E[u(t,x)^p]$$

we get the desired bound.

Spatial asymptotics

Work in progress with X. Chen, Y. Hu and S. Tindel

- For $t \geq 0$ fixed, we claim that

$$\lim_{R\to\infty} (\log R)^{-\frac{1}{1+H}} \log \left(\max_{|x|\leq R} u(t,x) \right) = C_H (t\mathcal{E})^{\frac{H}{1+H}},$$

almost surely, where

$$\mathcal{E} = \sup_g \left\{ \int_{\mathbb{R}} |\mathcal{F}g^2(\xi)|^2 |\xi|^{1-2H} d\xi - \frac{1}{2} \int_{\mathbb{R}} |g'(x)|^2 dx \right\},$$

and the supremum is over $\{g \in L^2(\mathbb{R}) : \|g\|_{L^2(\mathbb{R})} = 1, g' \in L^2(\mathbb{R})\}$.

- This means that $v(t,x) = \log u(t,x)$ (solution to the KPZ equation) satisfies

$$\lim_{R\to\infty} \frac{\max_{|x|\leq R} v(t,x)}{(\log R)^{\frac{1}{1+H}}} = C_H (t\mathcal{E})^{\frac{H}{1+H}}.$$

Note that $\frac{1}{1+H} = \frac{2}{3}$ if $H = \frac{1}{2}$.

Thanks for your attention!

Bon anniversaire pour Vlad!